ISS for TDS with point-wise dissipation and F-equivalence for general linear PDE: Applications

Epiphane LOKO

Ecole nationale des ponts et chaussées (CERMICS) & Université Paris Saclay (L2S)

Séminaire au LAGEPP, Université de Lyon

September 1, 2025

PART I

Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation

Joint work with
Antoine Chaillet, Yuan Wang, Iasson Karafyllis, and Pierdomenico Pepe

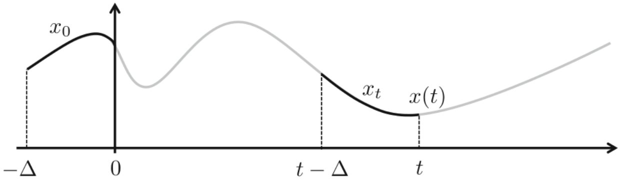
Time-delay system

Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \quad (1)$$

$u \in L^{\infty}_{loc}(\mathbb{R}^+, \mathbb{R}^m)$, $\Delta > 0$, $\quad x_t : [-\Delta, 0] \to \mathbb{R}^n$, $\quad x_t(s) = x(t+s)$.



f is Lipschitz on bounded sets and $f(0,0) = 0$, in what follows.

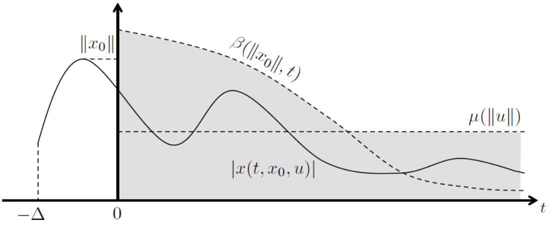
Input-to-state stability (ISS)

Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$ and all $u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$

$$|x(t, x_0, u)| \leq \max\{\beta(||x_0||, t), \mu(||u_{[0,t]}||)\}, \quad \forall t \geq 0.$$



Lyapunov-Krasovskii-Functional (LKF)

Definition 3

A functional $V : C([-\Delta,0],\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist $\underline{\alpha},\overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|). \tag{3}$$

$$|x(t)| := \left(\sum_{i=1}^{n} |x_i(t)|^2\right)^{1/2} \quad \text{and} \quad \|x_t\| := \sup_{\tau\in[-\Delta,0]} |x_t(\tau)|.$$

Dissipations of LKF

Definition 4

For TDS (1), an LKF V is said to be

1. an ISS LKF with LKF-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+ V \le -\alpha(V(x_t)) + \gamma(|u(t)|), \tag{4}$$

2. an ISS LKF with point-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+ V \le -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{5}$$

$D^+ V$ is the Driver's derivative of the functional V along the solution of (1).

General ISS characterization

Karafyllis, Pepe, and Jiang 2008

Theorem 5

TDS (1) *is ISS if and only if it admits an ISS LKF with LKF-wise dissipation.*

Is the point-wise dissipation enough to ensure ISS of TDS?

Chaillet, Pepe, et al. 2017

Conjecture 1

Assume that the system (1) *admits a LKF with a point-wise dissipation. Then it is ISS.*

In response,

1. Antoine Chaillet, Pierdomenico Pepe, et al. (2017). "Is a point-wise dissipation rate enough to show ISS for time-delay systems?" In: *IFAC-PapersOnLine* 50.1, pp. 14356–14361
2. Antoine Chaillet, Iasson Karafyllis, et al. (2023). "Growth conditions for global exponential stability and exp-ISS of time-delay systems under point-wise dissipation". In: *Systems & Control Letters* 178, p. 105570
3. Andrii Mironchenko et al. (2024). "ISS Lyapunov-Krasovskii theorem with point-wise dissipation: a V-stability approach". In: *2024 IEEE 63rd Conference on Decision and Control (CDC)*. IEEE, pp. 7896–7901

Chaillet, Pepe, et al. 2017

Conjecture 1

Assume that the system (1) *admits a LKF with a point-wise dissipation. Then it is ISS.*

In response,

1. Antoine Chaillet, Pierdomenico Pepe, et al. (2017). "Is a point-wise dissipation rate enough to show ISS for time-delay systems?" In: *IFAC-PapersOnLine* 50.1, pp. 14356–14361
2. Antoine Chaillet, Iasson Karafyllis, et al. (2023). "Growth conditions for global exponential stability and exp-ISS of time-delay systems under point-wise dissipation". In: *Systems & Control Letters* 178, p. 105570
3. Andrii Mironchenko et al. (2024). "ISS Lyapunov-Krasovskii theorem with point-wise dissipation: a V-stability approach". In: *2024 IEEE 63rd Conference on Decision and Control (CDC)*. IEEE, pp. 7896–7901

A way to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} V_2(x_t(s))ds, \tag{6}$$

which dissipates point-wisely as follow

$$D^+ V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding ke^{cs} in the kernel of integral part of (6), we get the LKF

$$W(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

Proposition 1

If $\alpha(|x(t)|) \geq pV_2(x(t))$, $(p > 0)$ then W is a LKF with LKF-wise dissipation and the TDS (1) is ISS.

Does this trick work systematically?

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{7}$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^{0} x_t(s)^4 ds,$$

$$W(x_t) := \frac{1}{4}x(t)^4 + \int_{-\Delta}^{0} ke^{cs} x_t(s)^4 ds.$$

Proposition 2

1. *System* (7) ***is ISS****.*
2. *V is an ISS LKF with* ***point-wise dissipation*** *for* (7)
3. *Given any $k, c > 0$, W is not a LKF with LKF-wise dissipation.*

ISS under point-wise dissipation

Theorem 6

Assume that there exists a LKF V which dissipates point-wisely as follow,

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \tag{8}$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{9}$$

Then, if

$$\liminf_{r \to +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0,$$

the system (1) *is ISS.*

Conclusions of Part I

1. The principle of adding exponential term in relaxed LKFs to make them strict, does not systematically work even for $1D$ TDS.
2. Growth condition is proposed to conclude ISS with relaxed LKF.
3. The proposed condition turns out to extend the existing ones.
4. The conjecture is still an open question.

PART II

Rapid stabilization of general linear systems with F-equivalence

Joint work with Amaury Hayat

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. We consider

$$\partial_t u = \mathcal{A}u + Bw, \tag{10}$$

where

- $\mathcal{A}$ and B are linear operators;
- $u(t, \cdot) \in X$ the state of the system;
- $w(t) \in \mathbb{C}$ the control input

Feedback stabilization

Conditions on $\mathcal{A}$ and B to:

1. Exhibit a feedback law $K \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$ s.t $w(t) = Ku(t, \cdot)$ and
$$\partial_t u = \mathcal{A}u + BKu, \tag{11}$$
is well-posed in X, i.e $\mathcal{A} + BK$ generates a C^0 semi-group in X;

2. show that any solution of (11) converges exponentially quick to 0 in X, i.e
$$\|u(t, \cdot)\|_X \leq ke^{-\mu t}\|u(0, \cdot)\|_X, \quad \forall t \geq 0.$$
equivalently, the semi-group generated by $\mathcal{A} + BK$ is exponentially stable in X.

F-equivalence method

Let $\lambda > 0$. Assume that $\mathcal{A}$ generates a **finite growth C^0 semi-group** and consider

$$\partial_t v = (\mathcal{A} - \lambda I)v. \tag{12}$$

There exists $\omega \in \mathbb{R}$, $k \geq 1$ such that

$$\|v(t,\cdot)\|_X \leq ke^{-(\lambda-\omega)t}\|v(0,\cdot)\|_X, \quad \forall t \geq 0.$$

If there exist an isomorphism T and feedback K s.t

$$\partial_t u = (\mathcal{A} + BK)u \quad \underset{v=Tu}{\Longleftrightarrow} \quad \partial_t v = (\mathcal{A} - \lambda I)v$$

then

$$\|u(t,\cdot)\|_X \leq Ke^{-(\lambda-\omega)t}\|u(0,\cdot)\|_X, \quad \forall t \geq 0. \tag{13}$$

F-equivalence method

Let $\lambda > 0$. Assume that $\mathcal{A}$ generates a **finite growth C^0 semi-group** and consider

$$\partial_t v = (\mathcal{A} - \lambda I)v. \tag{12}$$

There exists $\omega \in \mathbb{R}$, $k \geq 1$ such that

$$\|v(t,\cdot)\|_X \leq k e^{-(\lambda-\omega)t} \|v(0,\cdot)\|_X, \quad \forall t \geq 0.$$

If there exist an isomorphism T and feedback K s.t

$$\partial_t u = (\mathcal{A} + BK)u \quad \underset{v=Tu}{\Longleftrightarrow} \quad \partial_t v = (\mathcal{A} - \lambda I)v$$

then

$$\|u(t,\cdot)\|_X \leq K e^{-(\lambda-\omega)t} \|u(0,\cdot)\|_X, \quad \forall t \geq 0. \tag{13}$$

Fredholm transformation

- KdV equation: Coron and Lü 2014 and K–S equation: Coron and Lü 2015
- Boundary-controlled systems: Coron, Hu, and Olive 2017, Deutscher and Gabriel 2019, Redaud, Auriol, and Niculescu 2022
- The linear Schrodinger equation: Coron, Gagnon, and Morancey 2018
- Degenerate parabolic: Gagnon, Lissy, and Marx 2021 and Lissy and Moreno 2023
- The transport equation: Zhang 2022
- The linearized Saint-Venant system: Coron, Hayat, et al. 2022
- The 1D heat equation: Gagnon, Hayat, et al. 2022b
- The skew-adjoint system: Gagnon, Hayat, et al. 2022a

Question?

Conjecture 2

*Assume that the pair $(\mathcal{A}, B)$ is **exactly controllable and admissible** in a certain Hilbert space H. Then, for any $\lambda > 0$ there exists a unique isomorphism-feedback pair (T, K) such that T transforms the system*

$$\partial_t u = \mathcal{A}u + BKu$$

into the system

$$\partial_t u = \mathcal{A}u - \lambda u.$$

Answer in finite dimension

Coron 2015

Theorem 7

Consider a linear finite dimension system defined as

$$\dot{y} = Ay + Bz, \tag{14}$$

where $y \in \mathbb{R}^n$, $z \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$. *Assume that the pair* (A, B) *is controllable, i.e*

$$rank\{B, A^2 B, \cdots, A^{n-1} B\} = n. \tag{15}$$

Then, for any $\lambda > 0$, *there exists a unique isomorphism-feedback pair* $(T, K) \in GL(n, \mathbb{R}) \times \mathbb{R}^{1 \times n}$ *such that* $TB = B$ *and* T *transforms the system*

$$\dot{y} = Ay + BKy,$$

into the exponentially stable system

$$\dot{y} = Ay - \lambda y.$$

Answer in infinite dimension?

Existing results

- **When $\mathcal{A}$ is self-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2022b). "Fredholm transformation on Laplacian and rapid stabilization for the heat equation". In: *Journal of Functional Analysis* 283.12, p. 109664
- **When $\mathcal{A}$ is skew-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2022a). "Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves". In: *Annales de l'Institut Fourier*

Can we extend these results to any general "**spectral**" operator $\mathcal{A}$?

Answer in infinite dimension?

Existing results

- **When $\mathcal{A}$ is self-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2022b). "Fredholm transformation on Laplacian and rapid stabilization for the heat equation". In: *Journal of Functional Analysis* 283.12, p. 109664
- **When $\mathcal{A}$ is skew-adjoint:** Ludovick Gagnon, Amaury Hayat, et al. (2022a). "Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves". In: *Annales de l'Institut Fourier*

Can we extend these results to any general "**spectral**" operator $\mathcal{A}$?

Spectral assumptions on $\mathcal{A}$

Assume that

1. $\mathcal{A}$ generates a dissipative C^0 semi-group on a Hilbert space X.
2. The eigenvectors φ_n of $\mathcal{A}$ form a Riesz basis of X.
3. The eigenvalues λ_n have finite multiplicity and there exists $\alpha > 1$ such that

$$n^\alpha \lesssim |\lambda_n| + 1 \lesssim n^\alpha, \quad \forall n \in \mathbb{N}^*, \tag{16}$$

$$|\lambda_n - \lambda_p| \gtrsim n^{\alpha-1}|n-p|, \quad \forall n, p \in \mathbb{N}^*. \tag{17}$$

Main result

Theorem 8

Let $\gamma \in [0, (\alpha - 1)/2)$. *If* B *satisfies*

$$c_1 \leq |\langle B, \widetilde{\varphi_n} \rangle| \leq c_2 n^\gamma, \quad \forall n \in \mathbb{N}^*, \tag{18}$$

Then, there exist a feedback K *and an isomorphism* T *such that* T *maps the system*

$$\partial_t u = \mathcal{A}u + BKu \tag{19}$$

to the system

$$\partial_t v = \mathcal{A}v - \lambda v. \tag{20}$$

For any $\mu > 0$, (19) *is exponentially stable with decay rate* $\mu > 0$.

$(\widetilde{\varphi_n})_n$ is the bi-orthogonal family for $(\varphi_n)_n$ as $\langle \varphi_n, \widetilde{\varphi_m} \rangle = \delta_{nm}$.

Extended result

Theorem 9 (Rapid stabilization in X)

Let $\gamma \in [0, (\alpha - 1)/2)$, $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$, *and* $B \in \mathcal{H}^{-\alpha}$ *such that*

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \tag{21}$$

then for any $\mu > 0$, *there exists a (constructive) bounded linear feedback* K *such that the system* (11) *is exponentially stable in the study space* X *with decay rate* μ*.*

Relaxed controllability and admissibility

Assume that $\mathcal{A}$ is **skew-adjoint**. From Weiss and C.-Z. Xu 2011, Russell and Weiss 1994

Lemma 10

*If B is **admissible** and if the system is in addition **exactly controllable in** X, then*

$$c \leq |\langle B, \widetilde{\varphi_n} \rangle| \leq C, \; \forall\, n \in \mathbb{N}^*,$$

1. Admissibility and exact controllability in X imply (21) for $r = \gamma = 0$.
2. What if there is no exact controllability in X, i.e (21) with $r \in (1/2 - \alpha + \gamma, -\gamma)$?
   - Assumpion (21) $\quad \Rightarrow \quad |\langle B, \widetilde{\varphi_n} \rangle| \leq C n^{r+\gamma}$.
   - Exact controllability in X $\quad \Rightarrow \quad c \leq |\langle B, \widetilde{\varphi_n} \rangle|$.
   - $c \leq C n^{r+\gamma}$ is a contradiction since $r + \gamma < 0$.

What if there is no exact controllability in X?

The operator $\mathcal{A}$ is still assumed to be skew adjoint.

1. From Zabczyk 2020 and Trélat, Wang, and Y. Xu 2019, stabilization is not possible with $K \in \mathcal{L}(X, \mathbb{C})$.
2. From Theorem 9, stabilization is possible with feedback $K \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$.

Liu et al. 2022 and Ma, Wang, and Yu 2023 has already used $K \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$.

Conclusion of Part II

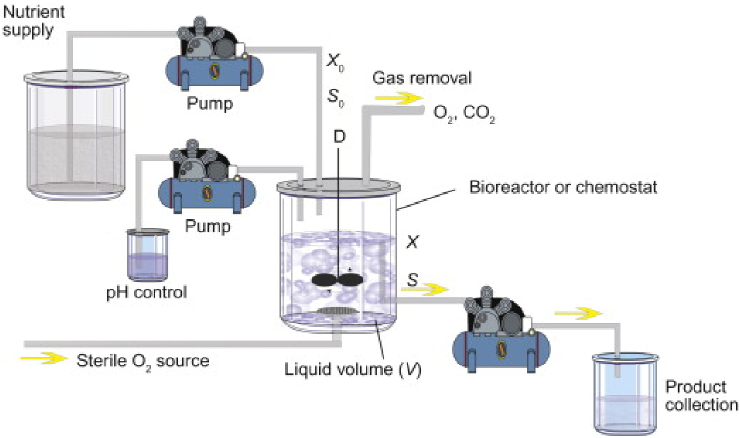
1. We provide more relaxed conditions to stabilize a general PDE
2. We weakened controllability and admissibility requirement for skew adjoint systems

PART III

The simplest chemostat model

Joint work with Iasson Karafyllis, Miroslav Krstic and Antoine Chaillet.

The chemostat: A microbial bioreactor



The chemostats are used to

1. produce antibiotics and some proteins
2. treat water (degradation of pollutant)
3. produce biomass
4. . . .

The simplest chemostat is modeled by:

$$\begin{cases} \dot{X} = (p_0\mu(S) - b - D)X \\ \dot{S} = D(S_{\text{in}} - S) - \mu(S)X. \end{cases} \tag{22}$$

Symbols	Amounts
X	Microbial concentration or biomass
S	Concentration of the substrate (nutrient)
S_{in}	Inlet concentration of the nutrient
$\mu(S)$	Specific growth rate
D	Dilution rate
b	Mortality rate
$p_0 = 1/k$	Yield factor

The parameters X, S, D, b, p_0 are positive and $\mu \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ with $\mu(0) = 0$.

A goal in the chemostat is to maintain the system in a non washout steady state.

Equilibrium point

An equilibrium point of the chemostat model (22) is (X^*, S^*) such that

$$p_0\mu(S^*) = b + D^*,$$
$$X^* = \frac{D^*(S_{\text{in}} - S^*)}{\mu(S^*)}.$$

Given $D^* > 0$, it may exist many equilibrium points

Stability

Given $D^* > 0$ and an equilibrium (X^*, S^*), the linearization of the system (22) is

$$\begin{pmatrix} \dot{X} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} 0 & p_0\mu'(S^*)X^* \\ -(b+D^*)/p_0 & -D^* - \mu'(S^*)X^* \end{pmatrix} \begin{pmatrix} X \\ S \end{pmatrix}. \tag{23}$$

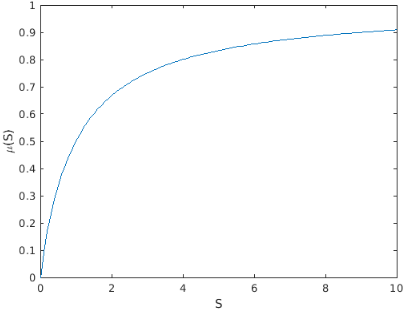
The characteristic polynomial is:

$$P(s) = s^2 + (D^* + \mu'(S^*)X^*)s + (b + D^*)\mu'(S^*)X^*. \tag{24}$$

The system is unstable if $\mu'(S^*) < 0$.

When considering the Monod kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S}, \tag{25}$$



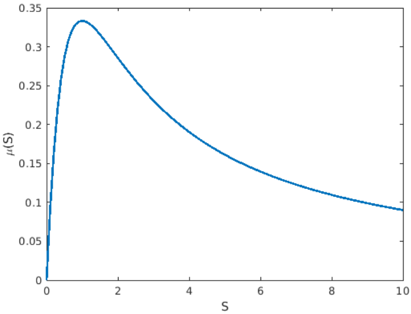
Dali-Youcef, Rapaport, and Sari 2022

Theorem 11

There exists a unique positive equilibrium point of (22) *which is globally asymptotically stable and locally exponentially stable.*

When, considering the Haldane kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S + aS^2}, \tag{26}$$



Theorem 12

The system (22) admits two positive equilibrium points (X^*, S^*), (X^{**}, S^{**}) *such that*

- (X^*, S^*) *is unstable.*
- (X^{**}, S^{**}) *is exponentially stable.*

Unstable open-loop chemostat model

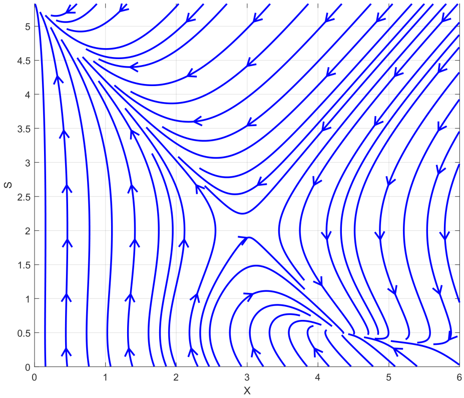


Figure: The phase diagram for the open-loop chemostat model (22) with Haldane equation

Theorem 13

Assume that

$$p_0\mu(S) > b, \quad \forall S \in [S^*, S_{in}]. \tag{27}$$

Then for every $\delta > 0$ *and* $\alpha \in [0,1)$ *the feedback law defined as*

$$D(X,S) = \frac{D^*\mu(S)X}{\mu(S^*)X^*} + \frac{\delta b}{(\mu(S^*))^{1+\alpha}} \begin{cases} |\mu(S) - \mu(S^*)|^{1+\alpha}, & \text{if} \quad S \leq S^* \\ 0, & \text{if} \quad S > S^* \end{cases} \tag{28}$$

achieves global asymptotic stabilization of (X^*, S^*). *Moreover, if* $\alpha > 0$ *then the feedback law* (28) *also achieves local exponential stabilization of* (X^*, S^*).

Stable closed-loop model

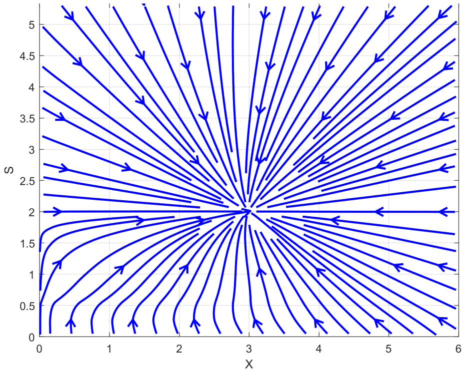


Figure: The phase diagram for the closed-loop chemostat model (22) with (28).

Is the sufficient condition (27) also necessary to ensure GAS of system (22)?

Theorem 14

Assume that there exists a constant $\overline{S} \in (S^*, S_{in})$ *such that*

$$p_0\mu(\overline{S}) < b \tag{29}$$

$$\mu'(S) \leq 0, \quad \forall S \in [\overline{S}, S_{in}]. \tag{30}$$

Then, there is no feedback law $D(X, S) \geq 0$ *that achieves global stabilization of the equilibrium* (X^*, S^*) *of* (22).

Perspectives

1. Allow the mortality b to be dependent of time t
2. Consider S_{in} and/or the volume V to be dependent of t
3. Consider another reproduction rate μ than Monod and Haldane

Thank you for your attention